

A Machian Request for the Equivalence Principle in Extended Gravity and non-geodesic motion

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Abstract

Starting from the origin of Einstein general relativity (GR) the request of Mach on the theory’s structure has been the core of the foundational debate. That problem is strictly connected with the nature of the mass-energy equivalence. It is well known that this is exactly the key point that Einstein used to realize a metric theory of gravitation having an unequalled beauty and elegance. On the other hand, the current requirements of particle physics and the open questions within extended gravity theories request a better understanding of Equivalence Principle (EP). The MOND theory by Milgrom proposes a modification of Newtonian dynamics and we consider a direct coupling between the Ricci curvature scalar and the matter Lagrangian showing that a non geodesic ratio m_i/m_g can be fixed and that Milgrom acceleration is retrieved at low energies.

1 Introduction

It is well known that *The Science of Mechanics* by Ernst Mach had a strong influence on Einstein and played an important role in the development of GR

[1]. In Newton's *Philosophiae Naturalis Principia Mathematica*, acceleration is considered as absolute. In the famous Gedankenexperiment of the rotating bucket filled with water, Newton deduced the existence of an absolute rotation by observing the curved surfaces on the water.

The aim of Newton was to explain the inertia through a sort of resistance to motion in the absolute space which, in this way, comes to be an agent and not a mere physical theater of coordinates, although unspecified. The first thinker to question the Newtonian reasoning was the philosopher George Berkeley in his *De Motu*, published in 1721 and he can be considered the precursor of Mach. Indeed, after more than 150 years, Mach proposed a radical criticism of Newton's absolute space and he concluded that the inertia would be an interaction that requires other bodies to manifest itself, so that it would make no sense in a Universe consisting of just a single mass. According to Mach, there is a total relational symmetry and every motion, uniform or accelerated, makes sense only in reference to other bodies. Therefore, following the so called *Mach Principle*, the inertia of a body is not an intrinsic property and depends on the mass distribution in the rest of the Universe. Einstein was very fascinated by Mach reasoning but it is widely acknowledged that Mach Principle is not fully incorporated into relativistic field equations [2]. The challenge of a *Machian physics* was accepted several times (though less than expected) in the context of both classical and quantum. Here we recall the Narlikar's theory with variable mass derived from Wheeler-Feynman-like action at a distance theory [3, 4]. Sciama's theory requires to get the inertia as "gravitational closeness" (and the perfect equivalence) under the precise cosmological condition $G\rho\frac{r^2}{c^2} = 1$ where r is the radius of the universe, ρ the density, c is the speed of light and G the gravitational constant [5]. In quantum contexts, and in Higgs times, the problem becomes more complex [6, 7, 8, 9, 10].

2 The Physics Under the Metric

Einstein has often stated that some Machian effects are present in GR. In particular, in the famous Lectures of 1921 [11] he states that it showed the following effects:

- 1) *The inertia of a body must increase when ponderable masses are piled up in its neighbourhood.*
- 2) *A body must experience an accelerating force when neighboring masses are accelerated and the force must be in the same direction as that acceleration.*
- 3) *A rotating hollow body must generate inside of itself a Coriolis field which deflects moving bodies in the sense of the rotation and a radial centrifugal field as well.*

Let us follow the Einstein reasoning. By considering the geodesic equation

$$\frac{d^2x_\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = 0 \quad (1)$$

Einstein worked in the weak- field approximation and the metric he found to represent the gravitational field due to a distribution of small masses corresponding to a density σ and having small velocities $\frac{dx_i}{ds}$ can be written as

$$\begin{aligned} g_{00} &= 1 - \frac{2G}{c^2} \int \frac{\sigma dV}{r} \\ g_{0i} &= \frac{4G}{c^2} \int \frac{dx_i}{ds} \frac{\sigma dV}{r} \\ g_{ij} &= -\delta_{ij} \left(1 + \frac{2G}{c^2} \int \frac{\sigma dV}{r} \right) \end{aligned} \quad (2)$$

The equation of motion in this field becomes

$$\frac{d}{dx^0} [(1 + \bar{\sigma}) v] = \nabla \bar{\sigma} + \frac{\partial A}{\partial x^0} + (\nabla \wedge A) \wedge v \quad (3)$$

where

$$\begin{aligned} \bar{\sigma} &\equiv \frac{G}{c^2} \int \frac{\sigma dV}{r} \\ A &= \frac{4G}{c^2} \int \frac{\sigma v dV}{r}. \end{aligned} \quad (4)$$

Einstein interpreted it by saying that the inertial mass is proportional to $1 + \bar{\sigma}$ and therefore increases when ponderable masses approach the test body

$$m_i = m_g \left(1 + \frac{G}{c^2} \int \frac{\bar{\sigma} dV}{r} \right) \quad (5)$$

Many physicists believe, according to C. Brans [12], that only the second and third effect are contained in GR. At first glance it seems that, if the Einstein interpretation is right, the EP is violated but we emphasize that all bodies with different inertial masses are still falling with the same acceleration in a gravitational field. In [13] the author analyzes what he calls *Modified Mach Principle* in the context of an expanding universe. He suggests the following definitions for the inertial mass within and beyond the bulge of galaxies as

$$\begin{aligned} m_i &= C & r &\leq R_0 \\ m_i &= \frac{C'}{r} = m_g \frac{R_0}{r} & r &> R_0 \end{aligned} \quad (6)$$

where C and C' are constants and he calls the first one as inertial mass versus gravitational interaction within the bulge, and the second one as inertial mass versus cosmological expansion beyond the bulge. It would seem that the introduction of a genuine Mach's principle implies a re-introduction of the distinction between inertial mass and gravitational mass, hidden under the metric of GR and the strong form of the EP. Let recall that the equivalence between m_i and m_g is the axiomatic and constructive keystone of GR. This raises the problem of the interpretation of the formalism able to establish the EP on the physical meaning of the relationship between m_i and m_g .

3 Inertial and Gravitational Mass

The nature of dark matter is one of the unsolved mysteries in cosmology since C. Zwicky measured the velocity dispersion of the Coma cluster of galaxies [14]. Let us rewrite the following relation

$$m_i \frac{v^2}{r} = \frac{GM_g m_g}{r^2}, \quad (7)$$

where m_i is a body that rotates around a gravitational mass M_g over a constant radius r . The relation

$$v = \sqrt[4]{GM_g a_0} \quad (8)$$

is in perfect agreement with the experimental data with a_0 is about $10^{-10} \frac{m}{s^2}$ [15] - [23]. Therefore we write

$$v^2 = \frac{GM_g}{r} \frac{m_g}{m_i} = \sqrt{GM_g a_0}. \quad (9)$$

It follows that

$$\frac{m_g}{m_i} = \sqrt{\frac{a_0 r^2}{GM_g}}. \quad (10)$$

If we do not interpret a_0 from the kinematic point of view but as a gravitational field, we can write

$$\frac{m_g}{m_i} = \sqrt{\frac{g_0}{g}}. \quad (11)$$

According to Mach and his interpreters, the inertial mass of a body arises as a consequence of its interactions with the Universe and so we assume that

$$\frac{m_g}{m_i} = \mu(x) \quad (12)$$

with $\mu = 1$ for $\left|\frac{g}{g_0}\right| \gg 1$ and $\mu = \sqrt{\frac{g}{g_0}}$ for $\left|\frac{g}{g_0}\right| \ll 1$.

A possible form of μ may be for example

$$\mu = \sqrt{\frac{g_0 + g}{g}} \quad (13)$$

where g is the field generated by nearby masses.

It is easy to verify that when $g \gg g_0$, circular velocity decreases in Keplerian way but when $g \ll g_0$ we obtain

$$v^2 = \frac{GM_g}{r} \sqrt{\frac{g_0}{g}} = GM_g \sqrt{\frac{g_0}{GM_g}} = \sqrt{GM_g g_0} \quad (14)$$

and finally

$$v = \sqrt[4]{GMg_0}. \quad (15)$$

Obviously the value of g_0 that fits all the data of galaxies rotation curves is about $10^{-10}m/s^2$. From the mathematical point of view the relations (8) and (15) coincide but from a physical point of view the situation is different. At every point in the Universe, the second Newtonian law is still valid even for small accelerations.

Deviations between inertial and gravitational mass as stressed by eq. (10) can have an intriguing geometrical explanation in the framework of $f(R)$ theories of gravity, assuming an explicit coupling between an arbitrary function of the scalar curvature, R , and the Lagrangian density of matter [25]. The most simple situation is to consider a weak modification of general relativity, which could be consistent with solar system tests, which implies a direct coupling between the Ricci curvature scalar and the matter Lagrangian [26]. Following [25, 26], let us consider the action (for the sake of simplicity we set $16\pi G = 1$, $c = 1$ and $\hbar = 1$ hereafter)

$$S = \int d^4x \sqrt{-g} (R + \lambda R \mathcal{L}_m + \mathcal{L}_m), \quad (16)$$

which only includes a coupling between the Ricci scalar and the matter Lagrangian, being λ the coupling constant, with respect to the well known canonical Einstein - Hilbert action of standard general relativity [27]

$$S = \int d^4x \sqrt{-g}. \quad (17)$$

Without loss of generality, we can also set $\lambda = 1$. Thus, the standard variation analysis in a local Lorentz frame enables to write [25, 26]

$$\begin{aligned} \delta \int d^4x \sqrt{-g} (R + R \mathcal{L}_m + \mathcal{L}_m) &= \int d^4x [\delta \sqrt{-g} (R + R \mathcal{L}_m + \mathcal{L}_m) + \sqrt{-g} \delta (R + R \mathcal{L}_m + \mathcal{L}_m)] \\ &= \int d^4x [\sqrt{-g} (1 + \mathcal{L}_m) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R + R \mathcal{L}_m + \mathcal{L}_m)] g^{\mu\nu} + d^4x \sqrt{-g} (1 + \mathcal{L}_m) g^{\mu\nu} \delta R_{\mu\nu}. \end{aligned} \quad (18)$$

The relation between the connections and the Ricci tensor [27] gives [25, 26]

$$g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} \partial_\alpha (\delta \Gamma_{\mu\nu}^\alpha) - g^{\mu\alpha} \partial_\alpha (\delta \Gamma_{\mu\nu}^\nu) \equiv \partial_\alpha X^\alpha, \quad (19)$$

where

$$X^\alpha \equiv g^{\mu\nu} (\delta \Gamma_{\mu\nu}^\alpha) - g^{\mu\alpha} (\delta \Gamma_{\mu\nu}^\nu). \quad (20)$$

Thus, the second integral in equation (18) results to be [25, 26]

$$\begin{aligned} \int d^4x \sqrt{-g} (1 + \mathcal{L}_m) g^{\mu\nu} \delta R_{\mu\nu} &= \int d^4x \sqrt{-g} (1 + \mathcal{L}_m) \partial_\alpha X^\alpha = \\ &= \int d^4x \partial_\alpha [\sqrt{-g} (1 + \mathcal{L}_m) X^\alpha] - \int d^4x \partial_\alpha [\sqrt{-g} (1 + \mathcal{L}_m)] X^\alpha. \end{aligned} \quad (21)$$

A standard assumption is that fields are equal to zero at infinity. Then we obtain [25, 26]

$$d^4x\sqrt{-g}(1+\mathcal{L}_m)g^{\mu\nu}\delta R_{\mu\nu} = - \int d^4x\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)]X^\alpha. \quad (22)$$

Let us calculate the quantity X^α . In a local Lorentz frame it is [25, 26]

$$\nabla_\beta g_{\mu\nu} = \partial_\beta g_{\mu\nu} = 0. \quad (23)$$

Thus, the well known definitions of the Christoffel connections [27] gives [25, 26]

$$\begin{aligned} \delta\Gamma_{\mu\nu}^\alpha &= \delta[\tfrac{1}{2}g^{\beta\alpha}(\partial_\mu \\ &= \tfrac{1}{2}g^{\beta\alpha}(\partial_\mu\delta_{\beta\nu} + \partial_\nu\delta_{\mu\beta} - \partial_\beta\delta_{\mu\nu}). \end{aligned} \quad (24)$$

In analogous way one gets [25, 26]

$$\delta\Gamma_{\mu\nu}^\nu = \tfrac{1}{2}g^{\nu\beta}\partial_\mu(\delta g_{\nu\beta}). \quad (25)$$

Using eqs. (24) and (25) we find [25, 26]

$$g^{\mu\nu}(\delta\Gamma_{\mu\nu}^\alpha) = \tfrac{1}{2}\partial^\alpha(g_{\mu\nu}\delta g^{\mu\nu}) - \partial^\mu(g_{\beta\mu}\delta g^{\nu\beta}) \quad (26)$$

and

$$g^{\mu\alpha}(\delta\Gamma_{\mu\nu}^\nu) = -\tfrac{1}{2}\partial^\alpha(g\delta g^{\nu\beta}). \quad (27)$$

Now, substituting in (20), we obtain [25, 26]

$$X^\alpha = \partial^\alpha(g_{\mu\nu}\delta g^{\mu\nu}) - \partial^\mu(g_{\mu\nu}\delta g^{\alpha\nu}). \quad (28)$$

In this way, equation (22) becomes [25, 26]

$$\begin{aligned} \int d^4x\sqrt{-g}(1+\mathcal{L}_m)g^{\mu\nu}\delta R_{\mu\nu} &= \\ &= \int d^4x\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)][\partial(g_{\mu\nu}\delta g^{\alpha\nu}) - \partial^\alpha(g\delta g^{\nu\beta})], \end{aligned} \quad (29)$$

which also gives [25, 26]

$$\begin{aligned} \int d^4x\sqrt{-g}(1+\mathcal{L}_m)g^{\mu\nu}\delta R_{\mu\nu} &= \\ &= \int d^4x\{g_{\mu\nu}\partial^\alpha\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)] - \int d^4x\{g_{\mu\nu}\partial^\mu\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)\delta g^{\alpha\nu}]\}. \end{aligned} \quad (30)$$

Inserting eq. (30) in the variation (18) we get

$$\begin{aligned} \delta \int d^4x\sqrt{-g}(R + R\mathcal{L}_m + \mathcal{L}_m) &= \int d^4x[\sqrt{-g}(1+\mathcal{L}_m)R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}(R + R\mathcal{L}_m + \mathcal{L}_m)]\delta g^{\mu\nu} + \\ \int d^4x\{g_{\mu\nu}\partial^\alpha\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)] - g_{\alpha\nu}\partial^\mu\partial_\alpha[\sqrt{-g}(1+\mathcal{L}_m)]\delta g_{\mu\nu}\} &+ d^4x\{(1+R)\delta(\sqrt{-g}\mathcal{L}_m)\}. \end{aligned} \quad (31)$$

This variation is equal to zero for [25, 26]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\mathcal{L}_m R_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu}\square) \mathcal{L}_m + \frac{(1+R)}{2}T_{\mu\nu}^{(m)} \quad (32)$$

which are the Einstein field equations modified by direct coupling between the Ricci curvature scalar and the matter Lagrangian. In fact, the standard stress-energy tensor [27]

$$T_{\mu\nu}^{(m)} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}}{\delta g^{\mu\nu}} \quad (33)$$

has been introduced in the modified field equations (32). Writing down explicitly the Einstein tensor and introducing a "total" stress-energy tensor [25, 26, 28]

$$T_{\mu\nu}^{(tot)} \equiv \frac{1}{(1+\mathcal{L}_m)} \left[(\nabla_\mu \nabla_\nu - g_{\mu\nu}\square) \mathcal{L}_m + \frac{(1+R)}{2}T_{\mu\nu}^{(m)} - \frac{R\mathcal{L}_m}{2}g_{\mu\nu} \right] \quad (34)$$

eqs. (32) can be put in the well known Einsteinian form

$$G_{\mu\nu} = \frac{1}{2}T_{\mu\nu}^{(tot)}, \quad (35)$$

in which a *curvature* contribution [28] is added and mixed to the *material* one. In other words, the high order terms contribute, like sources, to the modified field equations and have to be considered like *effective fields* ([28] for details). The condition of energy conservation [25, 26, 27]

$$\nabla^\mu G_{\mu\nu} = 0 \quad (36)$$

can be inserted in eqs. (35) and (34), obtaining

$$\nabla^\mu T_{\mu\nu}^{(m)} = \frac{1}{R+1}(g_{\mu\nu} \quad (37)$$

Now, we can introduce the well known stress-energy tensor of a perfect fluid [27]

$$T_{\mu\nu}^{(m)} \equiv (\epsilon + p)u_\mu u_\nu - pg_{\mu\nu}, \quad (38)$$

in order to test the motion of test particles [25, 26], where ϵ is the proper energy density, p the pressure and u_μ the fourth-velocity of the particles. This is the simplest version of a stress-energy tensor for the matter, concerning incoherent matter, and it is considered a good approximation in astrophysics frameworks [25, 26, 27]. Introducing the projector operator [25, 26]

$$P_{\mu\alpha} \equiv g_{\mu\alpha} - u_\mu u_\alpha, \quad (39)$$

we can apply the contraction $g^{\alpha\beta}P_{\mu\beta}$ to equation (37), obtaining [25, 26]

$$\frac{d^2 x_\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = F^\alpha \quad (40)$$

Thus, we find the existence of an extra force [25, 26]

$$F^\alpha \equiv (\epsilon + p)^{-1} P^{\alpha\nu} \left[\left(\frac{1}{R+1} \right) (\mathcal{L}_m + p) \nabla_\mu R + \nabla_\mu p \right] \quad (41)$$

showing that the motion of test particles is \emph{non-geodesic}. This extra force is orthogonal to the four-velocity of test masses [25, 26]

$$F^\alpha \frac{dx_\alpha}{ds} = 0. \quad (42)$$

The Newtonian limit in three dimensions of equation (40) reads [25, 26]

$$\vec{a}_{tot} = \vec{a}_n + \vec{a}_{ng}. \quad (43)$$

The total acceleration \vec{a}_{tot} is given by the ordinary Newtonian acceleration \vec{a}_n plus the repulsive acceleration \vec{a}_{ng} due to the extra (non-geodesic) force [25, 26]. Eq. (43) and a bit of three-dimensional geometry permit to write the Newtonian acceleration \vec{a}_n as [25, 26]

$$\vec{a}_n = \frac{1}{2} (a_{tot}^2 - a_n^2 - a_{ng}^2) \frac{\vec{a}_{tot}}{a_{ng} a_{tot}} \quad (44)$$

Considering the limit in which \vec{a}_{ng} dominates (i.e. $a_n \ll a_{tot}$) one gets

$$a_n \simeq \frac{a_{tot} \vec{a}_{tot}}{2a_{ng}} \left(1 - \frac{a_{ng}^2}{a_{tot}^2} \right). \quad (45)$$

Thus, the extra aceleration is given by [25, 26]

$$a_0 \equiv \left[\frac{1}{2a_{ng}} \left(1 - \frac{a_{ng}^2}{a_{tot}^2} \right) \right]^{-1}, \quad (46)$$

and combining eq. (46) with eqs. (10) and (11) one gets

$$\frac{m_g}{m_i} = \sqrt{\frac{g_0}{g}} = \sqrt{\frac{r^2}{\frac{M_g}{2a_{ng}} \left(1 - \frac{a_{ng}^2}{a_{tot}^2} \right)}} \quad (47)$$

with

$$g_0 = \frac{r^2}{\frac{1}{2a_{ng}} \left(1 - \frac{a_{ng}^2}{a_{tot}^2} \right)}. \quad (48)$$

Thus, we have shown that in our model the ratio between gravitational and inertial mass is explained in an elegant, geometric way through a direct coupling between the Ricci curvature scalar and the matter Lagrangian which generates a non geodesic motion of test particles.

4 Conclusions

The assumption of an R -dependent inertial mass is in accordance with the spirit of Mach principle and Einstein himself tried to implement this hypothesis in the context of General Relativity [29]. The possible relation m_g/m_i is deduced by comparing it with Milgrom's rotational equation that is in perfect agreement with the experimental data. However, the interpretation given here is different, and it leaves unchanged in any point the second law of dynamics. Finally we have shown that the ratio between gravitational and inertial mass is explained in geometric way through a direct coupling between the Ricci curvature scalar and the matter Lagrangian which generates a non geodesic motion of test particles. Such a non geodesic motion is due to the presence of the centrifugal extra force in eq. (42).

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